Understanding the Partial Derivative

The example presented here shows how we can interpret the Partial Derivatives

$$\frac{\partial z}{\partial x} = f_x(x_0, y_0) \quad \text{and} \quad \frac{\partial z}{\partial y} = f_y(x_0, y_0) \quad \text{as}$$

the rate of change of $z = f(x, y)$ as the input (x, y) changes from (x_0, y_0) in a manner which leaves the value of all but one variable fixed and constant.

Consider the function $z = f(x, y) = 10 + 5x - 4y$. $\frac{\partial z}{\partial x} = +5$ and $\frac{\partial z}{\partial y} = -4$.

With input $(x_0, y_0) = (6, 3)$, $f(6, 3) = 28 = 10 + 30 - 12$.

How does the value of z change from 28 when we change the input (x, y) from $(6, 3)$?

TWO VIEWS (Restricted by the Fixing of a Single Variable)

$z = f(6,3) = 28$

Increase x from 6 but leave y = 3 (Fix y)

	$z = f(6.0, 3) = 28.0$
h = 0.1	$z = f(6.1, 3) = 28.5$
h = 0.5	$z = f(6.5, 3) = 30.5$
h = 1.0	$z = f(7.0, 3) = 33.0$

The value $z = f(x, y)$ increases 5 times as much as x increases.

$$f_x(6,3) = \lim_{h \to 0} \frac{f(6+h, 3) - f(6,3)}{h} = +5$$

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} = +5$$

$z = f(6,3) = 28$

Increase y from 3 but leave x = 6 (Fix x)

	$z = f(6, 3.0) = 28.0$
h = 0.1	$z = f(6, 3.1) = 27.6$
h = 0.5	$z = f(6, 3.5) = 26.0$
h = 1.0	$z = f(6, 4.0) = 24.0$

The value $z = f(x, y)$ decreases 4 times as much as y increases.

$$f_y(6,3) = \lim_{h \to 0} \frac{f(6, 3+h) - f(6,3)}{h} = -4$$

$$\frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h} = -4$$

It was easy to see that the situation described above was certain to happen because the function

$z = f(x, y) = 10 + 5x - 4y$ was so simple.

What would happen if the function were more complicated, like $z = x^3 y^2 - \cos(x^2 + y^3)$?

That might be too complicated at this point, so let us consider a simpler function.

Consider the function $z = f(x, y) = x^2 y^3 - y^4$. $\dfrac{\partial z}{\partial x} = 2xy^3$ and $\dfrac{\partial z}{\partial y} = 3x^2y^2 - 4y^3$.

With input $(x_0, y_0) = (3, 1)$, $f(3, 1) = 8 = 9 - 1$.

How does the value of z change from 8 when we change the input (x, y) from $(3, 1)$?

TWO VIEWS (Restricted by the Fixing of a Single Variable)

$z = f(3,1) = 8$

Increase x from 3 but leave $y = 1$ (Fix y)

	$z = f(3.000, 1) = 8.0000$
$h = 0.001$	$z = f(3.001, 1) = 8.006001$
$h = 0.01$	$z = f(3.01, 1) = 8.0601$
$h = 0.1$	$z = f(3.1, 1) = 8.61$

The value $z = f(x, y)$ increases about 6 times as much as x increases.

$$f_x(3,1) = \lim_{h \to 0} \frac{f(3+h, 1) - f(3,1)}{h} = +6$$

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} = 2xy^3$$

At $(x,y) = (3,1)$, $\dfrac{\partial z}{\partial x} = f_x(3,1) = 6$

$z = f(3,1) = 8$

Increase y from 1 but leave $x = 3$ (Fix x)

$z = f(3, 1.000) = 8.00000$
$z = f(3, 1.001) = 8.023021$
$z = f(3, 1.01) = 8.232105$
$z = f(3, 1.1) = 10.5149$

The value $z = f(x, y)$ increases about 23 times as much as y increases.

$$f_y(3,1) = \lim_{h \to 0} \frac{f(3, 1+h) - f(3,1)}{h} = +23$$

$$\frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h} = 3x^2y^2 - 4y^3$$

At $(x,y) = (3,1)$, $\dfrac{\partial z}{\partial y} = f_y(3,1) = 23$